

5.1 Let $\mathcal{M}$ be a differentiable manifold and ∇ a connection on $\mathcal{M}$.

- (a) Show that there exists no $(1,2)$ -type tensor field A on $\mathcal{M}$ with the property that, in any local coordinate system $(x^1, \dots, x^n)$ on $\mathcal{M}$

$$A_{ij}^k = \Gamma_{ij}^k.$$

Hint: Check how Γ_{ij}^k transforms under changes of coordinates.

- (b) Show that the torsion $T : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{M})$ of the connection ∇ , which is defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

is a tensor field.

- (c) Let $\bar{\nabla}$ be a (possibly) different connection on $\mathcal{M}$. Show that the difference $\nabla - \bar{\nabla} : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{M})$ is also a tensor field. Deduce that, there exists a $(1,2)$ -type tensor field A such that, in any given local coordinate system $(x^1, \dots, x^n)$,

$$A_{ij}^k = \Gamma_{ij}^k - \bar{\Gamma}_{ij}^k$$

where Γ_{ij}^k and $\bar{\Gamma}_{ij}^k$ are the Christoffel symbols of ∇ and $\bar{\nabla}$, respectively.

5.2 Let $\Psi : \mathcal{M}^n \rightarrow \mathbb{R}^{n+1}$ be an immersion such that $\Psi(\mathcal{M})$ is a *spacelike* hypersurface of $(\mathbb{R}^{n+1}, \eta)$ and let $\bar{g} = \Psi_*\eta$ be the induced metric. Let $(x^1, \dots, x^n)$ be a local coordinate chart on $\mathcal{M}$. Compute the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection associated to $\bar{g}$ in the $(x^1, \dots, x^n)$ coordinates as functions of Ψ and its derivatives.

5.3 Let $\mathcal{M}$ be a smooth manifold equipped with a connection ∇ . We can extend the connection ∇ to a map $\nabla : \Gamma(\mathcal{M}) \times \text{Ten}_l^k(\mathcal{M}) \rightarrow \text{Ten}_l^k(\mathcal{M})$ by the requirements that

- ∇ satisfies the Leibniz rule with respect to tensor products, i.e. for all $X \in \Gamma(\mathcal{M})$

$$\nabla_X(f \otimes g) = \nabla_X f \otimes g + f \otimes \nabla_X g,$$

- ∇ commutes with contractions, i.e.

$$\nabla_X(\text{tr} A) = \text{tr}(\nabla_X A).$$

Show that, in any local coordinate chart $(x^1, \dots, x^n)$, if Γ_{ij}^k are the Christoffel symbols of ∇ then, for every 1-form ω :

$$(\nabla_{\frac{\partial}{\partial x^i}} \omega)_j = \partial_i \omega_j - \Gamma_{ij}^k \omega_k.$$

Moreover, for any (k, l) -tensor field T :

$$\begin{aligned} (\nabla_{\frac{\partial}{\partial x^a}} T)^{i_1 \dots i_k}_{j_1 \dots j_l} &= \partial_a T^{i_1 \dots i_k}_{j_1 \dots j_l} + \Gamma_{ab}^{i_1} T^{b i_2 \dots i_k}_{j_1 \dots j_l} + \dots + \Gamma_{ab}^{i_k} T^{i_1 \dots i_{k-1} b}_{j_1 \dots j_l} \\ &\quad - \Gamma_{a j_1}^b T^{i_1 i_2 \dots i_k}_{b j_2 \dots j_l} - \dots - \Gamma_{a j_l}^b T^{i_1 \dots i_{k-1} b}_{j_1 \dots j_{l-1} b}. \end{aligned}$$

5.4 Let $(\overline{\mathcal{M}}, \bar{g})$ be a *Riemannian* manifold (i.e. $\bar{g}$ is positive definite) and let us define the Lorentzian manifold $(\mathcal{M}, g)$ so that $\mathcal{M} = \mathbb{R} \times \overline{\mathcal{M}}$ and g is the product metric $g = -(dt)^2 + \bar{g}$; this means that, for every local coordinate chart $(x^1, \dots, x^n)$ on $\mathcal{U} \subset \overline{\mathcal{M}}$, if we extend it to a local coordinate chart $(t, x^1, \dots, x^n)$ on $\mathbb{R} \times \mathcal{U} \subset \mathcal{M}$ so that t is simply the projection on the $\mathbb{R}$ factor, then

$$g = -dt^2 + \bar{g}_{ij} dx^i dx^j.$$

Show that a curve $\gamma : (0, 1) \rightarrow (\mathcal{M}, g)$ is a geodesic (for the Levi-Civita connection of g) if and only, in any local coordinate system $(t; x^1, \dots, x^n)$ as above, if it can be written in the form

$$\gamma(s) = (t(s); \bar{\gamma}^i(s))$$

where $t(s) = \lambda_1 s + \lambda_0$ for some $\lambda_1, \lambda_0 \in \mathbb{R}$ and $\bar{\gamma} : (0, 1) \rightarrow \overline{\mathcal{M}}$ is a geodesic of $(\overline{\mathcal{M}}, \bar{g})$.

5.5 In this exercise, we will prove that there exist compact Lorentzian manifolds which are **geodesically incomplete** (recall that, as a consequence of the Hopf–Rinow theorem in Riemannian geometry, every compact Riemannian manifold is geodesically complete). Consider the manifold $\mathcal{M} = \mathbb{R}^2 \setminus 0$ equipped with the metric

$$g = \frac{1}{u^2 + v^2} du dv.$$

- (a) Verify that $(\mathcal{M}, g)$ is a smooth Lorentzian manifold and that the map $(u, v) \rightarrow (\lambda \cdot u, \lambda \cdot v)$ is an isometry for every $\lambda \neq 0$.
- (b) Consider the group of isometries $\Gamma = \{(u, v) \rightarrow (2^k u, 2^k v), k \in \mathbb{Z}\}$. Show that the quotient space $\mathcal{M}/\Gamma$ is a compact manifold. Show also that $\mathcal{M}/\Gamma$ inherits a natural metric $\tilde{g}$ from $(\mathcal{M}, g)$ so that the quotient map $(\mathcal{M}, g) \rightarrow (\mathcal{M}/\Gamma, \tilde{g})$ is a local isometry.
- (c) Show that the map $(\mathcal{M}, g) \rightarrow (\mathcal{M}/\Gamma, \tilde{g})$ maps geodesics to geodesics. Compute the geodesic equation on $(\mathcal{M}, g)$ and deduce that $(\mathcal{M}/\Gamma, \tilde{g})$ contains a geodesic $\gamma : (a, b) \rightarrow \mathcal{M}/\Gamma$ with $b < +\infty$ which *cannot* be extended beyond $t = b$.